On the Relationship between Sum-Product Networks and Bayesian Networks

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Introduction

- We prove that every Sum-Product Network (SPN) can be converted into a Bayesian Network (BN) in linear time and space.
- The generated BN has a simple directed bipartite graphical structure with Algebraic Decision Diagrams (ADDs) as representations of the local probability distributions.
- Applying the Variable Elimination algorithm (VE) to the generated BN will recover the original SPN.
- We introduce *normal* SPN and present a theoretical analysis of the consistency and decomposability properties.

Background

Definition (Poon and Domingos):

- A rooted DAG with indicator variables as leaves and sum nodes, product nodes as internal nodes.
- Edges from sum nodes are associated with nonnegative weights.
- The value of a product node is the product of the values of its children. The value of a sum node is the weighted sum of the values of its children. The value of an SPN is the value of its root.

Scope: The set of variables that have indicators among the node's descendants.

Complete: An SPN is *complete* iff each sum node has children with the same scope.

Consistent: An SPN is consistent iff no variable appears negated in one child of a product node and non-negated in another.

Decomposable: An SPN is decomposable iff for every product node v, $\text{scope}(v_i) \cap \text{scope}(v_j) = \varnothing$ where $v_i, v_j \in Ch(v), i \neq j$.

Algebraic Decision Diagram: A graphical representation a real function with boolean input variables, where the graph is a rooted DAG.

X_1	X_2	X_3	X_4	$f(\cdot)$	X_1	X_2	X_3	X_4	$f(\cdot)$
0	0	0	0	0.4	1	0	0	0	0.4
0	0	0	1	0.6	1	0	0	1	0.6
0	0	1	0	0.3	1	0	1	0	0.3
0	0	1	1	0.3	1	0	1	1	0.3
0	1	0	0	0.4	1	1	0	0	0.1
0	1	0	1	0.6	1	1	0	1	0.1
0	1	1	0	0.3	1	1	1	0	0.1
0	1	1	1	0.3	1	1	1	1	0.1

Figure 1: Tabular representation

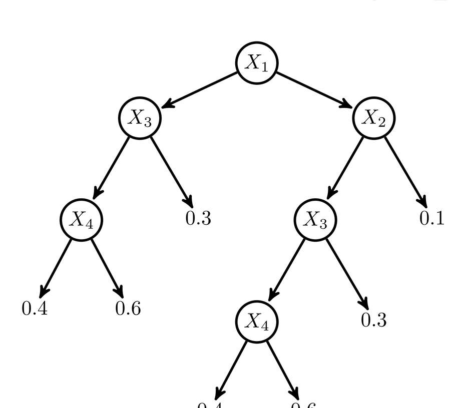


Figure 2: Decision tree representation

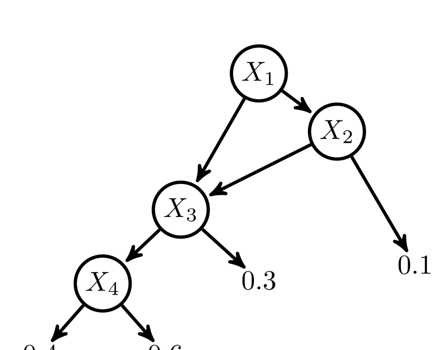


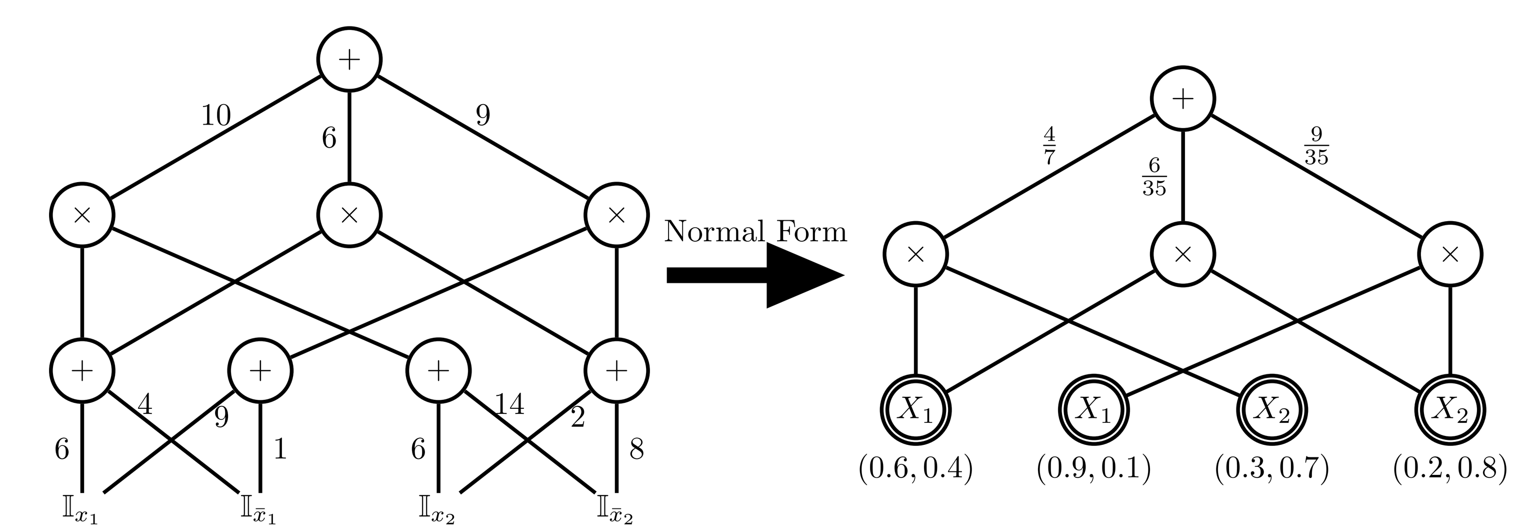
Figure 3: ADD representation

Our Results

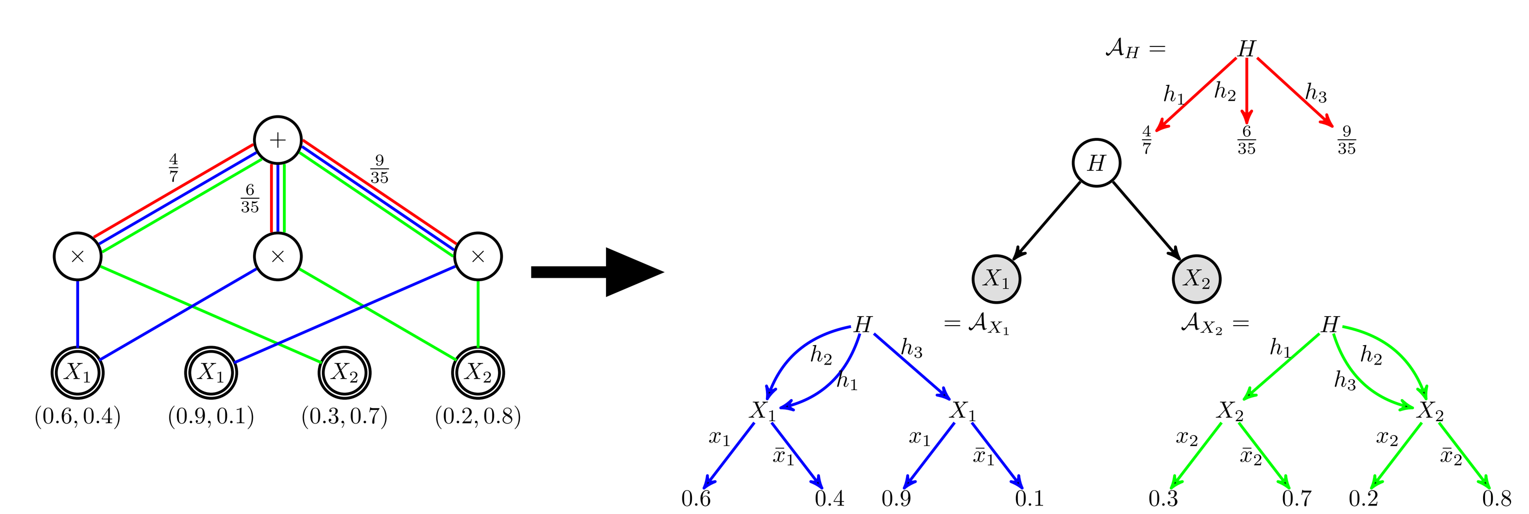
Normal SPN:

- Complete and decomposable.
- Locally normalized weights.

Theorem 1:
For any complete and consistent SPN $\mathcal{S}$, there exists a normal SPN $\mathcal{S}'$ such that $\Pr_{\mathcal{S}}(\cdot) = \Pr_{\mathcal{S}'}(\cdot)$ and $|\mathcal{S}'| = O(|\mathcal{S}|^2)$.



Theorem 2 (SPN to BN):
There exists an algorithm that converts any complete and decomposable SPN $\mathcal{S}$ over Boolean variables $\mathbf{X}_{1:N}$ into a BN $\mathcal{B}$ with CPDs represented by ADDs in time $O(N|\mathcal{S}|)$. Furthermore, $\mathcal{S}$ and $\mathcal{B}$ represent the same distribution and $|\mathcal{B}| = O(N|\mathcal{S}|)$.

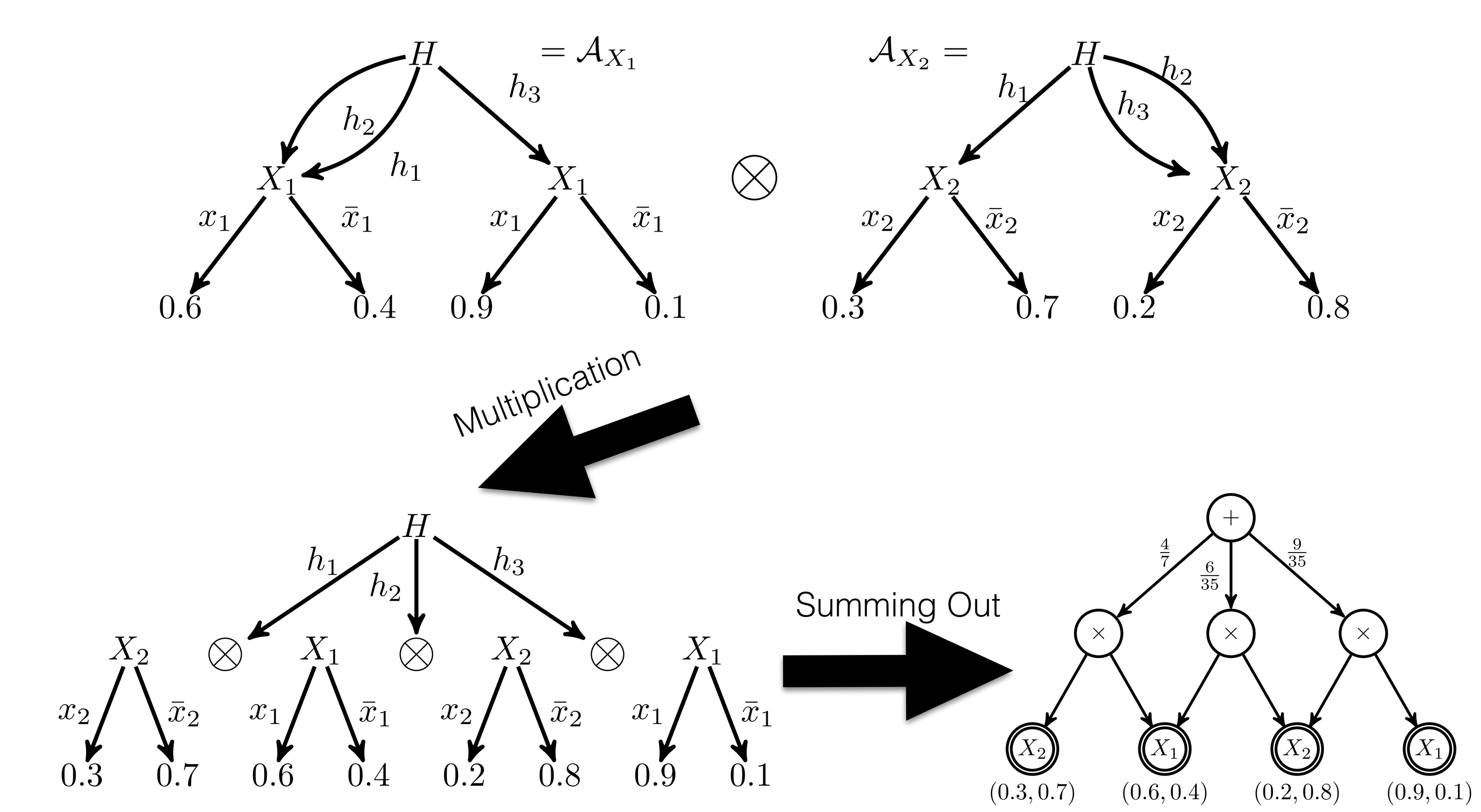


Corollary 3:
There exists an algorithm that converts any complete and consistent SPN $\mathcal{S}$ over Boolean variables $\mathbf{X}_{1:N}$ into a BN $\mathcal{B}$ with CPDs represented by ADDs in time $O(N|\mathcal{S}|^2)$. Furthermore, $\mathcal{S}$ and $\mathcal{B}$ represent the same distribution and $|\mathcal{B}| = O(N|\mathcal{S}|^2)$.

Remark 4:
The BN $\mathcal{B}$ generated from $\mathcal{S}$ has a simple bipartite DAG structure, where all the source nodes are hidden variables and the terminal nodes are the Boolean variables $\mathbf{X}_{1:N}$.

Remark 5:
Assuming sum nodes alternate with product nodes in SPN $\mathcal{S}$, the depth of $\mathcal{S}$ is proportional to the maximum in-degree of the nodes in $\mathcal{B}$, which, as a result, is proportional to a lower bound of the tree-width of $\mathcal{B}$.

Theorem 6 (BN to SPN):
Given the BN $\mathcal{B}$ with ADD representation of CPDs generated from a complete and decomposable SPN $\mathcal{S}$ over Boolean variables $\mathbf{X}_{1:N}$, the original SPN $\mathcal{S}$ can be recovered by applying the Variable Elimination algorithm to $\mathcal{B}$ in $O(N|\mathcal{S}|)$.



Remark 7:
The combination of the above two theorems shows that distributions for which SPNs allow a compact representation and efficient inference, BNs with ADDs also allow a compact representation and efficient inference.

Conclusion

- The CSI among variables helps to reduce the inference complexity to enable efficient exact inference even for graphical models with large tree-width.
- There may exist other techniques to convert an SPN into a BN with a more compact representation and also a smaller tree-width.
- Structure and parameter learning for SPNs can benefit from the Bayesian network perspective.
- The analysis showed in this paper can be straightforwardly applied to analyze the relationship between SPNs and Markov networks.